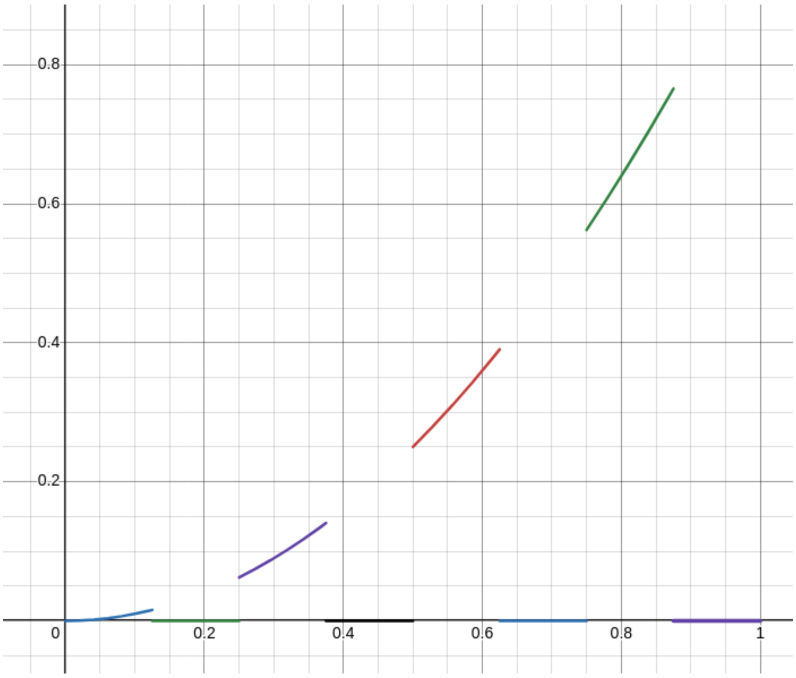
1. The set $\{x(t) \in L^1[0,1] : 0 \leq x(t) \leq t^2\}$ is not compact.

Proof: let's construct a sequence of functions f_n, such that there aren't any converging subsequences.

On the n-th step we will break the closed interval $[0,1]$ into 2^n parts. On the first interval we will assign $f_n = t^2$, on the second interval $f_n = 0$, on the third interval $f_n = t^2$ and so on. The resulting function is indeed integratable.



If we consider $|f_n(t) - f_m(t)|, n \neq m$ then it will be equal to t^2 on half of the smaller intervals and 0 on the other half, and so

$$\int_0^1 |f_n(t) - f_m(t)|\ dt = \frac{1}{2}\int_0^1 t^2 dt = \frac{1}{6}$$

And so there is the same positive difference between each ellement of the sequence and hence it can't have any converging subsequences ■.

2. Suppose we have two Lie algegras on $\mathbb{R}^3$: g^+ and g^- with brackets $[-,-]_+$ and $[-,-]_+$ such that

$$[e_1, e_2]_+ = e_3 \qquad [e_1, e_2]_- = e_3$$

$$[e_2, e_3]_+ = e_1 \qquad [e_2, e_3]_- = e_1$$

$$[e_3, e_1]_+ = e_2 \qquad [e_3, e_1]_- = -e_2$$

Are they in the same orgit of $GL(3, \mathbb{R})$ action?

Solution no, they aren't. Let's construct Killing form from these two algebras. Reminder that Killing form is a biliniar form:

$$B(x, y) = \operatorname{tr}(ad(x) \circ ad(y))$$

where $ad(x)$ is a linear operator $[x, -]$. This biliniar form is constructed canonically from the Lie algebra. Let's calculate it for our two algebras:

$$B_+(x, y) = -2x_1y_1 - 2x_2y_2 - 2x_3y_3$$

$$B_-(x, y) = 2x_1y_1 - 2x_2y_2 + 2x_3y_3$$

And so as we can see these two forms have different signatures and hence one can't be obtained from another using $GL(3, \mathbb{R})$ group action, which means that one lie algebra also can't be obtained from the other one using $GL(3, \mathbb{R})$ group action ■.